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A TEMPORAL FINITE ELEMENT METHOD FOR THE DYNAMIC ANALYSIS OF FLEXIBLE MECHANISMS

Y. Wang and Z. Wang

Department of Mechanical Engineering, University of Maryland, College Park, MD 20742, U.S.A.

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1. INTRODUCTION

Vibration analysis of high speed and light weight mechanism systems must consider the mechanisms as elastic bodies in order to accurately predict their performance of specified functions. A general model to describe the elastic motion of a mechanism can be properly established with the use of standard spatial finite element methods, which results in a set of second order differential equations. A common assumption in this procedure is that the total motion is comprised of an elastic motion superposed onto the rigid body motion. As a result, the equations of elastic motion have as an important feature time dependent coefficients. If the effects of non-linear elastic deflections and/or non-linear joint characteristics (e.g., clearances) are considered, the equations of motion will be non-linear [1].

The problem of a numerical solution for the formulated differential equations has been the focus of extensive research in the field of dynamics of mechanisms [2]. The dynamic response of a flexible mechanism can be viewed as a transient response and a steady state response. A class of methods for the dynamic analysis is based on the classical numerical integration techniques such as the fourth order Runge–Kutta method and the Newmark method [1, 2]. These methods can be made general and flexible for an initial value problem. They are, however, inefficient for the purpose of steady state response.

Another class of methods were developed for the linear systems. In an algorithm presented in reference [3], the time domain is divided into a series of subintervals, in which the coefficient matrices are approximated to be constant. Within each subinterval, a modal analysis approach can be applied, and the transient response can be found step by step. A closed form solution procedure is also devised for the steady state response by applying the boundary conditions for periodic motions and by solving a set of linear algebraic equations [4]. This algorithm is further modified in reference [5] with improved computational efficiency. A similar algorithm for steady state response is also proposed in reference [1] based on a multi-step integration scheme. This class of methods cannot be directly applied to solve the non-linear problems.

This paper presents a new algorithm for solving the general dynamic response of flexible mechanisms. The algorithm is in the same spirit of time domain discretization as of reference [3], but it uses an entirely different scheme. The proposed method is based on a variational approach and employs temporal finite elements. The governing differential equations of motion with time-varying coefficient matrices are first transformed into a weighted integral form in an application of Hamilton's principle. Then the equations of the weak form are discretized in the time domain by applying the finite element in time method. The final set of algebraic equations are obtained, describing system response in terms of a set of temporal nodes of all spatial degrees of freedom of the system.

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The finite element in time (FET) method offers several potential advantages. Because the solution is sought by propagation through the time elements, the resulting algebraic system is block-diagonal and therefore can be solved efficiently. Solution procedures are formulated to obtain both transient and steady state solutions. Stability of the system can be readily determined by applying Floquet's theory, since the required transition matrix is a byproduct of the solution procedure. In addition, non-linearities can be included in the FET formulation for general non-linear problems.

2. MOTION OF FLEXIBLE MECHANISMS

A linkage mechanism generally consists of links which are connected by joints (revolute or prismatic) and permits large motion between the links (Figure 1). The links are usually considered to be rigid if they operate at low speeds. At higher speeds, effects of mass distribution and elasticity become significant. For such mechanisms, design procedures must account for the elastic motion of links. The analytical model used in a design must include the mass, stiffness, and damping characteristics, the external loading, and the characteristics of the joints. The analytical models do not admit closed form solutions so approximate solutions must be obtained by a numerical technique such as the spatial finite element method [2, 3, 6–8]. The literature in this field is huge and a representative set of the works are reviewed in references [2, 9, 10].

The spatial finite element method has become a standard technique for analysis of a flexible mechanism. To describe it briefly, the absolute motion of each link is decomposed into the *rigid body motion* upon which is superimposed the *elastic motion*. The rigid body motion is treated as being governed only by the dynamics of the ideal rigid link, while the elastic motion is measured relative to a moving co-ordinate system fixed in the rigid link to account for elastic effects [9, 6]. For a large class of elastic mechanisms such as a four-bar crank rocker with a large flywheel at the crank, experimental investigations have validated that the rigid link motion is not influenced by the elastic motion [6, 11]. Assuming the driving link to be moving with a constant angular velocity ω, the dynamics of the mechanism can be described by a set of second order ordinary differential equations [2, 6, 9]

$$\mathbf{M}(t)\ddot{\mathbf{u}} + \mathbf{C}(t)\dot{\mathbf{u}} + \mathbf{K}(t)\mathbf{u} = \mathbf{f}(t), \tag{1}$$

where $\mathbf{u}$ denotes the elastic motions of all the spatial degrees of freedom (m), and $\mathbf{f}$ is the forcing vector containing all rigid body inertia forces and externally applied forces and movements $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ are time-varying dynamic matrices. The equations are linear with time-dependent coefficients and loading. This is the class of mechanisms to be discussed in this paper.

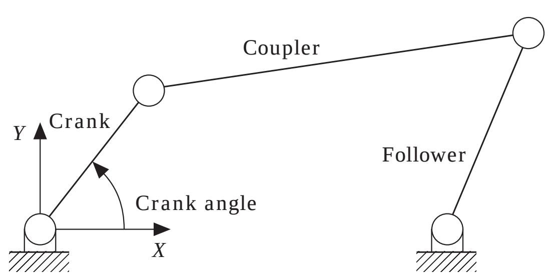

Figure 1. An elastic four-bar linkage mechanism.

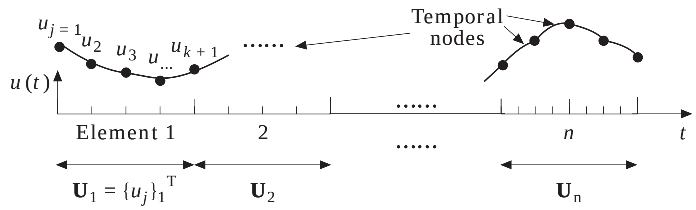

Figure 2. Time finite elements and temporal nodes.

3. THE TEMPORAL FINITE ELEMENT METHOD

The equations of motion (1) are said to be in explicit form due to the expression for absolute acceleration. If one wishes to determine an approximate solution of the equations, a weak formulation based on weighted integral statements of the equations may be derived and a variational method can be used for approximation [12]. This would avoid the complexities associated with the explicit form. The principle of virtual work, or Hamilton's principle, is one such weak formulation. There has been an extensive discussion in the literature concerning the use of a weak form as an alternative to numerical solutions of dynamics problems [13, 14].

The Hamilton's weak principle is expressed in the virtual energy form

$$\int_{t_I}^{t_F} \delta\mathbf{u}^{\mathrm{T}}(\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} - \mathbf{f})\,\mathrm{d}t = 0. \tag{2}$$

Integrating the first term of the integrand by parts gives the reduced form

$$\int_{t_I}^{t_F} \{\delta\dot{\mathbf{u}}^{\mathrm{T}}(\mathbf{M}\dot{\mathbf{u}}) + \delta\mathbf{u}^{\mathrm{T}}[-\mathbf{C}\dot{\mathbf{u}} - \mathbf{K}\mathbf{u}] + \delta\mathbf{u}^{\mathrm{T}}\mathbf{f}\}\,\mathrm{d}t = \delta\mathbf{u}^{\mathrm{T}}\mathbf{p}|_{t_I}^{t_F}, \tag{3}$$

where $\mathbf{p} = \mathbf{M}\dot{\mathbf{u}}$ defines the set of momenta. This equation of weak form describes the real motion of the system between the two known time instants t_I and t_F. Here, $\mathbf{p}$ appears only as a discrete quantity at the ends of the time interval, allowing it to be found without differentiation of $\mathbf{u}$. This equation is said to be in displacement form because it only involves the variation of $\mathbf{u}$ [13, 15].

The use of the finite element method is the most powerful tool for an easy approximation of the variational equation (3). For the analysis, the time interval $T = t_F - t_I$ is divided into a finite number (n) of time elements, usually equally spaced for convenience, as illustrated in Figure 2. Similar to the standard finite element technique in spatial discretization, a set of nodal displacements $\mathbf{U}_i$ for each time element i are defined and the displacement $\mathbf{u}$ and velocity $\dot{\mathbf{u}}$ within the element are interpolated among the nodal displacements $\mathbf{U}_i$ by using standard shape functions, for example, the Lagrange's polynomials, as

$$\mathbf{u}(t) = \mathbf{N}_i(\xi)\mathbf{U}_i \quad \text{and} \quad \dot{\mathbf{u}}(t) = \dot{\mathbf{N}}_i(\xi)\mathbf{U}_i, \qquad (\xi = (t - t_i)/(t_{i+1} - t_i), i = 1, 2, \dots, n), \tag{4}$$

where the $(m \times m(k+1))$ matrix $\mathbf{N}$ for the elemental shape function has $(k+1)$ components of polynomial function if the order of interpolation is chosen to be k. That is,

$$\mathbf{U}_i = \{\mathbf{u}_1^{\mathrm{T}}, \dots, \mathbf{u}_{k+1}^{\mathrm{T}}\}_i^{\mathrm{T}}, \qquad \mathbf{N} = [N_1\mathbf{I}, \dots, N_{k+1}\mathbf{I}], \qquad \dot{\mathbf{N}} = [\dot{N}_1\mathbf{I}, \dots, \dot{N}_{k+1}\mathbf{I}], \quad (5\text{–}7)$$

where $\mathbf{I}$ is an identity matrix having size m, the total spatial degrees of freedom of the system. Correspondingly, the nodal displacement vector $\mathbf{U}_i$ has $(k+1)$ temporal components for each spatial degree of freedom and for each time element. The Lagrange family of interpolation functions is used in this study.

Thus, the equation of motion (3) can be put in the following varitional form:

$$\sum_{i=1}^{n} \int_{t_i}^{t_{i+1}} \delta \mathbf{U}_i^{\mathrm{T}} \{[\dot{\mathbf{N}}_i^{\mathrm{T}} \mathbf{M} \dot{\mathbf{N}}_i - \mathbf{N}_i^{\mathrm{T}} \mathbf{C} \dot{\mathbf{N}}_i - \mathbf{N}_i^{\mathrm{T}} \mathbf{K} \mathbf{N}_i] \mathbf{U}_i + \mathbf{N}_i^{\mathrm{T}} \mathbf{f}\} \, \mathrm{d}t = \delta \mathbf{U}_i^{\mathrm{T}} \mathbf{B}_i, \tag{8}$$

where $\mathbf{B}_i$ is a vector of $(k+1)$ elements containing linear momenta of the dynamic system at the two ends of the time element, $\mathbf{B}_i = \{-\mathbf{p}^{\mathrm{T}}(t_i), 0, \ldots, 0, \mathbf{p}^{\mathrm{T}}(t_{i+1})\}^{\mathrm{T}}$. Since $\delta \mathbf{U}_i$ is completely arbitrary, the finite element approximation at the element level is given by

$$\mathbf{A}_i \mathbf{U}_i + \mathbf{P}_i = \mathbf{B}_i, \tag{9}$$

where

$$\mathbf{A}_i = \int_{t_i}^{t_{i+1}} \dot{\mathbf{N}}_i^{\mathrm{T}} \mathbf{M}(t) \dot{\mathbf{N}}_i \, \mathrm{d}t - \int_{t_i}^{t_{i+1}} \mathbf{N}_i^{\mathrm{T}} \mathbf{C}(t) \dot{\mathbf{N}}_i \, \mathrm{d}t - \int_{t_i}^{t_{i+1}} \mathbf{N}_i^{\mathrm{T}} \mathbf{K}(t) \mathbf{N}_i \, \mathrm{d}t, \tag{10}$$

$$\mathbf{P}_i = \int_{t_i}^{t_{i+1}} \mathbf{N}_i^{\mathrm{T}} \mathbf{f}(t) \, \mathrm{d}t, \qquad \mathbf{B}_i = \{-\mathbf{p}^{\mathrm{T}}(t_i), 0, \ldots, 0, \mathbf{p}^{\mathrm{T}}(t_{i+1})\}^{\mathrm{T}}. \tag{11, 12}$$

Thus, the finite element approximation results in a set of linear algebraic equations in $m \times (k+1)$ unknown nodal displacement vectors for each time element.

4. SOLUTION PROCEDURES

The finite element method in the time domain has the crucial property that the dynamic response of a system is sought through propagation forward in time. As a result, the nodal displacement vectors can be solved effectively. In the case of either an initial value problem or a steady state response, the finite element approximation to the equation of motion can be obtained element by element without actually assembling the elements. The solution procedures are described as follows.

4.1. *Solution of initial value problems*

In the case of an initial value problem, the initial displacement and momentum vectors of the first time element are prescribed, i.e., $\mathbf{u}(t_1) = \mathbf{u}(0)$ and $\mathbf{p}(t_1) = \mathbf{p}(0) = \mathbf{M}\dot{\mathbf{u}}(0)$. The transient solution requires a propagation of the initial displacement and momentum vectors forward in time to the end of the first time element and so forth, finally to the end of the last time element. The propagation for the first element is achieved by solving for $\mathbf{u}(t_2)$ and $\mathbf{p}(t_2)$ from the set of linear algebraic equations (9) and the result is conveniently described as

$$\begin{Bmatrix} \mathbf{u}(t_2) \\ \mathbf{p}(t_2) \end{Bmatrix} = \mathbf{T}_1 \begin{Bmatrix} \mathbf{u}(t_1) \\ \mathbf{p}(t_1) \end{Bmatrix}. \tag{13}$$

As a simple matter, this propagation is carried through to element i, yielding the transient response described by

$$\begin{Bmatrix} \mathbf{u}(t_{i+1}) \\ \mathbf{p}(t_{i+1}) \end{Bmatrix} = [\mathbf{T}_i \mathbf{T}_{i-1} \cdots \mathbf{T}_1] \begin{Bmatrix} \mathbf{u}(t_1) \\ \mathbf{p}(t_1) \end{Bmatrix}. \tag{14}$$

4.2. *Steady state solution*

For the steady state solution, the entire time period of interest would equal the period τ of the steady state response sought. That is, $t_F = t_I + \tau$. For example, $\tau = 2\pi/\omega$, for the fundamental periodic response. The periodic boundary conditions require both displacements and momenta to be identical at t_1 and t_{n+1}, or $\mathbf{u}(t_1) = \mathbf{u}(t_{n+1})$ and $\mathbf{p}(t_1) = \mathbf{p}(t_{n+1})$. This would result in dropping out the boundary term of the right side of the variational equation (9) when the time elements are assembled to form the system level solution equations [13, 14, 15].

Since the dynamic response is sought through propagation forward in time, the steady state solution can be obtained without actual assembly of the global system equations. This property is similar to that of finite elements in space used in solid mechanics. There exists a standard procedure of static condensation in the space finite element method, by which the elemental degrees of freedom that are not involved in satisfying the inter-element compatibility are eliminated before the element is assembled into the overall system equations, yielding less unknowns in the system equations [12]. A similar procedure for finite elements in time is described in reference [16] for obtaining the solution of linear equations efficiently. It is briefly discussed as follows.

Consider the tangent matrix $\mathbf{A}_i$, the displacement, force, and momentum vectors, $\mathbf{U}_i$, $\mathbf{P}_i$, and $\mathbf{B}_i$ of linear equation (9) of element i, to be partitioned as

$$\left[\begin{array}{c|c} \mathbf{A}_i^{bb} & \mathbf{A}_i^{bi} \\ \hline \mathbf{A}_i^{ib} & \mathbf{A}_i^{ii} \end{array}\right] \begin{Bmatrix} \mathbf{U}_i^b \\ \hline \mathbf{U}_i^i \end{Bmatrix} + \begin{Bmatrix} \mathbf{P}_i^b \\ \hline \mathbf{P}_i^i \end{Bmatrix} = \begin{Bmatrix} \mathbf{B}_i^b \\ \hline \mathbf{0} \end{Bmatrix}, \tag{15}$$

where superscripts b and i correspond to the temporal nodes at the ends and in the interior of the current time element separately, and $\mathbf{B}_i^b = \{-\mathbf{p}^{\mathrm{T}}(t_i), \mathbf{p}^{\mathrm{T}}(t_{i+1})\}^{\mathrm{T}}$. Then, the lower part of the partitioned equation can be solved for $\mathbf{U}_i$ as

$$\mathbf{U}_i^i = -(\mathbf{A}_i^{ii})^{-1}\mathbf{P}_i^i - (\mathbf{A}_i^{ii})^{-1}\mathbf{A}_i^{ib}\mathbf{U}_i^b. \tag{16}$$

When this equation is substituted into the upper part of partitioned equation (15) for $\mathbf{U}_i^i$ and the terms are collected, the remaining term for $\mathbf{U}_i^b$ can be written as

$$\bar{\mathbf{A}}_i^{bb}\mathbf{U}_i^b + \bar{\mathbf{P}}_i^b = \mathbf{B}_i^b, \tag{17}$$

where the reduced tangent matrix and force vector are given by

$$\bar{\mathbf{A}}_i^{bb} = \mathbf{A}_i^{bb} - \mathbf{A}_i^{bi}(\mathbf{A}_i^{ii})^{-1}\mathbf{A}_i^{ib}, \qquad \bar{\mathbf{P}}_i^b = \mathbf{P}_i^b - \mathbf{A}_i^{bi}(\mathbf{A}_i^{ii})^{-1}\mathbf{P}_i^i. \tag{18, 19}$$

When the next element is generated, the condensation procedure can be applied identically, resulting in a similar set of reduced equations. These two sets of reduced equations are further condensed by imposing compatibility conditions of the displacement vector and the boundary conditions at the common node of the two elements, yielding the elimination of the common node. The above procedure is repeated for each remaining element, and the final set of equations can be written in terms of the initial and final time nodes $\mathbf{U}_I$ and $\mathbf{U}_F$ only. Finally, when the periodic conditions $\mathbf{U}_I = \mathbf{U}_F$ are imposed for the periodic steady state response, the initial and final nodal displacements can be solved.

This procedure is the same as Gaussian elimination. Once the boundary nodes have been computed, then the condensed unknown nodes can be recovered by a back substitution process for each corresponding condensation. The procedure is continued for all nodes in the system, allowing a complete solution of the equations to be obtained.

5. STABILITY OF STEADY STATE SOLUTION

Stability of any periodic steady state solution obtained above is investigated by applying Floquet's theory to determine the eigenvalues of the transition matrix that relates the initial and final states. This transition matrix is a byproduct of the initial value solution (14), i.e.,

$$\begin{Bmatrix} \mathbf{u}(t_F) \\ \mathbf{p}(t_F) \end{Bmatrix} = [\mathbf{T}_n \mathbf{T}_{n-1} \cdots \mathbf{T}_1] \begin{Bmatrix} \mathbf{u}(t_I) \\ \mathbf{p}(t_I) \end{Bmatrix}. \tag{20}$$

Therefore, there is no need for special effort for calculating the transition matrix.

6. EXAMPLE

A four-bar linkage mechanism is used for illustration (Figure 1). The crank of the mechanism is rotated with a constant angular velocity $\omega = 35$ rad/s. All links are considered to be elastic. The spatial finite element model was gernerated using typical Hermite beam elements with shear deformation. For convenience, each link was modelled with two beam elements of equal length. There is a total of 18 spatial degrees of freedom. The forms of mass and stiffness matrices of the beam element can be found in, for example, reference [6].

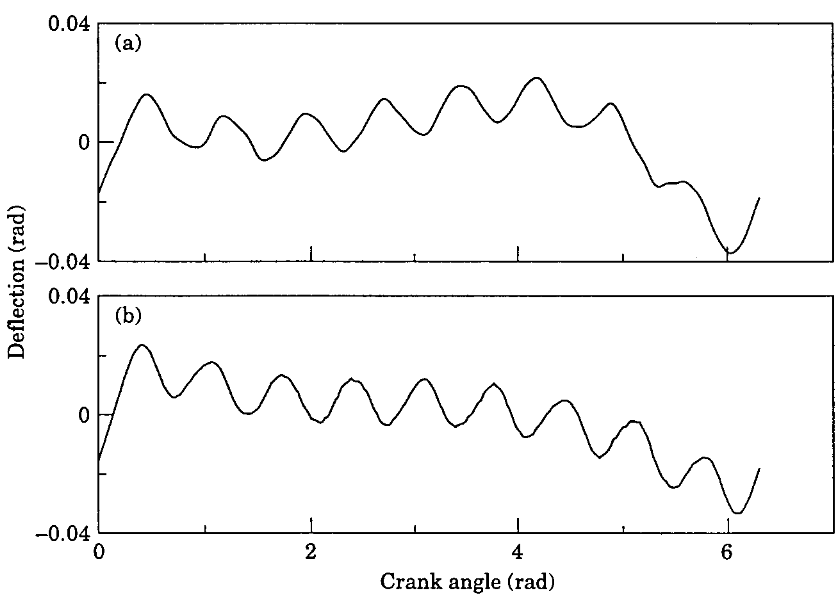

Figure 3. Angular deflection at (a) the left end of the coupler, (b) the fixed pivot of the follower.

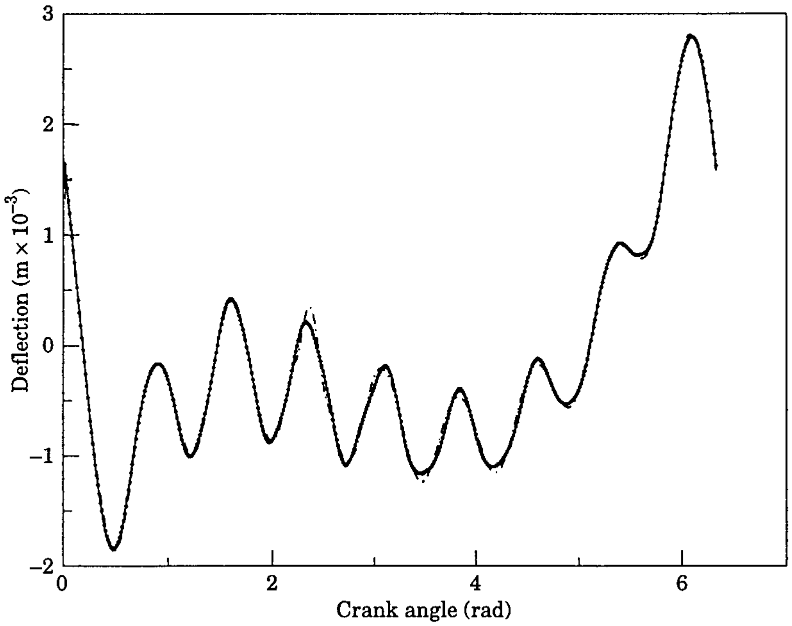

Figure 4. Horizontal deflection at the center of the coupler: –·–·, $n = 20$; · · · , $n = 40$; —, $n=50$.

Fifty time elements are used for the four-bar mechanism with a fourth order Lagrangian interpolation function. The fundamental steady state solution is investigated. The fundamental period $\tau = 2\pi/\omega$, corresponding to a cycle of the crank rotation, was divided into fifty equally spaced subintervals. Proportional damping was assumed for each link and the damping ratio was set to be 0·02.

Angular deflection at the left end of the coupler is plotted in Figure 3(a). The angular steady state response at the fixed pivot of the follower is plotted in Figure 3(b). As seen from the figure, the elastic deflection of the links are significantly influenced by the vibration resonance phenomena [6, 5].

Figure 4 shows the steady state response at the center of the coupler in the horizontal direction, where the deflection is computed with various numbers of time elements, $n = 20$, 40, and 50 respectively. These curves show that the finite element in time approximation solution converges at $n = 50$.

7. CONCLUSION

The transient and steady state motions of elastic mechanisms are studied with a finite element method in both the space and the time domains. The conventional spatial finite element method leads to a linear time-varying model describing the system responses. The temporal finite element method transforms the resulting dynamic equations into a set of linear algebraic equations with block-diagonal coefficient matrices. The finite element method in the time domain is based on a Hamilton's weak principle, paralleling the variational methods in elastostatics. This proposed method does not require the modal analysis technique, and the resulting equations can be solved step by step through propagation forward in time. An example is included to illustrate the procedures applied to a four-bar linkage.

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REFERENCES

1. X. GAO 1983 *Modern Kinematics: Developments in the Last Forty Years* (A. G. Erdman, editor), 387–394. New York: John Wiley. Solution methods for dynamic response of flexible mechanisms.
2. G. N. SANDOR and A. G. ERDMAN 1984 *Advanced Mechanism Design*. Englewood Cliffs, N.J.: Prentice-Hall.
3. A. MIDHA, A. G. ERDMAN and D. A. FROHRIB 1979 *Journal of Mechanical Design* **101**, 138–148. A computationally efficient numerical algorithm for the transient response of high-speed elastic linkages.
4. A. MIDHA, A. G. ERDMAN and D. A. FROHRIB 1979 *Journal of Mechanical Design* **101**, 154–162. A closed-form numerical algorithm for the periodic response of high-speed elastic linkages.
5. Z. YANG and J. P. SADLER 1993 *Journal of Mechanical Design* **115**, 848–855. A numerical efficient algorithm for steady-state response of flexible mechanism systems.
6. D. A. TURCIC and A. MIDHA 1984 *Journal of Dynamic Systems, Measurement, and Control* **106**, 249–260. Dynamic analysis of elastic mechanisms. Part 1 and 2.
7. C. Y. LIAO and C. K. SUNG 1990 *Flexible Mechanism, Dynamics and Robot Trajectory* (S. Derby, editor) DE-24, 503–510. New York: ASME. An elastodynamic analysis and control of the flexible linkages using piezoceramic sensors and actuators.
8. S. NAGARAJAN and D. A. TURCIC 1990 *Journal of Dynamic Systems, Measurement, and Control* **112**, 203–214. Lagrangian formulation of the equations of motion for elastic mechanisms with mutual dependence between rigid body and elastic motions. Part 1.
9. B. S. THOMPSON and C. K. SUNG 1986 *Mechanisms and Machine Theory* **21**, 351–359. A survey of finite element techniques for mechanism design.
10. G. G. LOWEN and C. CHASSAPIS 1986 *Mechanisms and Machine Theory* **21**, 33–42. The elastic behavior of linkages: An update.
11. F. W. LIOU and A. G. ERDMAN 1989 *Journal of Vibration, Acoustics, Stress, and Reliability in Design* **111**, 35–41. Analysis of high-speed flexible four-bar linkage: Part 1.
12. K.-J. BATHE 1982 *Finite Element Procedures in Engineering Analysis*. Englewood Cliffs, N.J.: Prentice-Hall.
13. M. BORRI, G. GHIRINGHELL, M. LANZ, P. MANTEGAZZA and T. MERLINI 1985 *Computers and Structures* **20**, 495–508. Dynamic response of mechanical systems by a weak Hamiltonian formulation.
14. D. A. PETERS and A. P. IZADPANAH 1988 *Computational Mechanics* **3**, 73–88. Hp-version finite elements for the space-time domain.
15. M. BORRI, C. BOTTASSO and P. MANTEGAZZA 1992 *Meccanica* **27**, 119–130. Basic features of the time finite element approach for dynamics.
16. M. BORRI 1986 *Computers and Mathematics with Applications* **12A**, 149–160. Helicopter rotor dynamics by finite element time approximation.